Problem 9

If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a}[f(x)g(x)]$.

Solution

Suppose that

$$\begin{cases} \lim\limits_{x\to a}[f(x)+g(x)] = 2 \\ \lim\limits_{x\to a}[f(x)-g(x)] = 1 \end{cases}.$$

The limit of a sum is the sum of the limits, and the limit of a difference is the difference of the limits.

$$\begin{cases} \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = 2 \\ \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = 1 \end{cases}$$

Set $B = \lim\limits_{x\to a} f(x)$ and $C = \lim\limits_{x\to a} g(x)$ for convenience.

$$\begin{cases} B + C = 2 \\ B - C = 1 \end{cases}$$

This is a system of two equations for two unknowns that can be solved. Add the respective sides to eliminate C.

$$2B = 3$$

$$B = \frac{3}{2}$$

Subtract the respective sides to eliminate B.

$$2C = 1$$

$$C = \frac{1}{2}$$

This means

$$\begin{cases} \lim\limits_{x\to a} f(x) = \dfrac{3}{2} \\ \lim\limits_{x\to a} g(x) = \dfrac{1}{2} \end{cases}.$$

Therefore, since the limit of a product is the product of the limits,

$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}.$$